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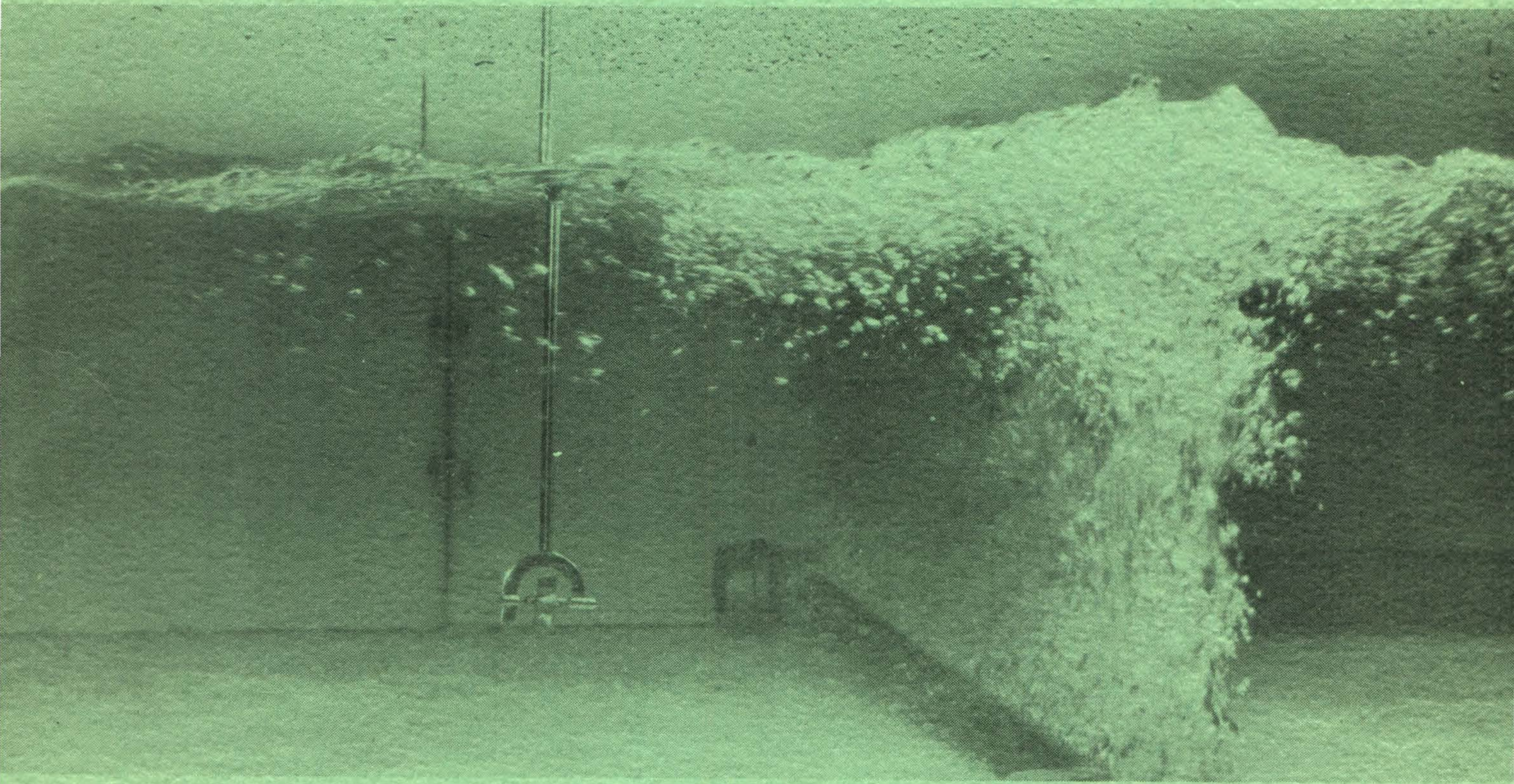
IN A SHALLOW COMPRESSIBLE LIQUID



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GRAVITATIONAL WAVES IN A SHALLOW COMPRESSIBLE LIQUID

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GRAVITATIONAL WAVES IN A SHALLOW COMPRESSIBLE LIQUID

This paper develops the theory of the propagation of waves through a horizontally stratified compressible liquid under the influence of gravity, with two simplifying assumptions: (1) The wave amplitudes are small, so that a linear or first-order theory suffices; (2) the liquid is assumed to be shallow so that the lengths of all waves are large in comparison with the liquid depth.

THE WAVE EQUATION

The general equations of motion for a compressible liquid are

force equation: $$-\overrightarrow{\text{grad}}\ p = \rho\left[\frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \overrightarrow{\text{grad}})\ \vec{U}\right] - \vec{F}, \tag{1}$$

continuity equation: $$\text{div}(\rho\vec{U}) = -\frac{\partial \rho}{\partial t}, \tag{2}$$

equation of state: $$\rho = \rho\ (p). \tag{3}$$

Here, p, ρ and $\vec{U}$ are the pressure, density and velocity of the liquid, all functions of the three space coordinates (x,y,z) and the time (t), and $\vec{F}$ is the externally applied body force per unit volume. We choose axes so that the waves are propagated in the x direction, and the y axis is vertically upward. Then none of the quantities depends on z, $\vec{F}$ is in the y direction and equal to $-\rho g$, and of the three components (u,v,w) of $\vec{U}$, $w = 0$.

In the absence of waves, it is assumed that the liquid density is a specified function of y only: $\rho_0(y)$. Then when waves are present, ρ can be written in the form $\rho = \rho\ (y) + \rho_1(x,y,t)$. For small amplitude waves ρ_1 is small in comparison with ρ_0, and is said to be a quantity of first order, while ρ_0 is said to be a quantity of zero order. In a similar way, we can write $p = p_0(y) + p_1(x,y,t)$, where the zero order term p_0 is determined below from ρ_0, and p_1 is of first order. The form of the equation of state (3) may depend on y, but is assumed not to depend on x or z. Then the quantity $d\rho/dp$ also depends only on y; as is well known, it is equal to $1/c^2$, where c(y) is the speed of pure elastic waves in the liquid.[1] We therefore have, to first order:

$$p_1 = \rho_1 c^2 \tag{4}$$

The non-vanishing components u,v of $\vec{U}$ are different from zero only when there is wave motion, so that they are of first order. Thus the term $\rho\ (\vec{U} \cdot \overrightarrow{\text{grad}})\vec{U}$ can be omitted from (1), since it is of second order. Equations (1) and (2) then become:

[1] Coulson, C. A., "Waves", pp. 87–89 (Interscience, New York, 1947).

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial u}{\partial t}, \tag{5}$$

$$-\frac{\partial p}{\partial y} = \rho \frac{\partial v}{\partial t} + \rho g, \tag{6}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = -\frac{\partial \rho}{\partial t}. \tag{7}$$

All of the terms in Eqs. (5) and (7) are of first or higher order, so that they can be written, to first order:

$$-\frac{\partial p_1}{\partial x} = \rho_o \frac{\partial u}{\partial t}, \tag{8}$$

$$\rho_o \frac{\partial u}{\partial x} + \frac{\partial(\rho_o v)}{\partial y} = -\frac{\partial \rho_1}{\partial t}. \tag{9}$$

Eq. (6), on the other hand, has both zero and first order parts. This equation can easily be integrated with respect to y. We say that the free surface, which is assumed to be at $y = 0$ when no waves are present, is at $y = \eta\,(x,t)$, where η is a first order quantity. Then if the ambient pressure above the free surface of the liquid is p_{oo}, Eq. (6) integrates to

$$p = p_{oo} + \int_y^{\eta} (\rho g + \rho \frac{\partial v}{\partial t})\, dy. \tag{10}$$

When Eq. (10) is written out through terms of first order, and use is made of (4), it becomes

$$p_o + p_1 = p_{oo} + \int_y^o \rho_o g dy + \int_o^{\eta} \rho_o g dy + \int_y^o \left(\frac{g p_1}{c^2} + \rho_o \frac{\partial v}{\partial t}\right) dy. \tag{11}$$

The first term on the left side and the first two terms on the right side are evidently of zero order, and the others of first order. Since the zero order terms are independent of the wave motion, they can always be equated. Thus (11) breaks up into two equations:

$$p_o = p_{oo} + \int_y^o \rho_o g dy \tag{12}$$

$$p_1 = \rho_{oo} g \eta + \int_y^o \frac{g p_1}{c^2} dy + \int_y^o \rho_o \frac{\partial v}{\partial t} dy; \tag{13}$$

the integral $\int_o^{\eta} \rho_o g\, dy$ has been replaced, to first order, by $\rho_{oo} g \eta$, where ρ_{oo} is the surface density $\rho_0(0)$. Eq. (12) is the expected expression for the static pressure as a function of depth.

We now assume that the second integral on the right side of (13) can be neglected in comparison with the other terms. It will be shown below that this is justified if the liquid is shallow, that is, if the depth h of the liquid is small in comparison with all wave lengths of interest. We thus replace Eq. (13) by

$$p_1 = \rho_{oo} g\eta + \int_y^o \frac{gp_1}{c^2}\, dy. \tag{14}$$

It is easily verified by direct substitution that the desired solution of the integral Eq. (14) is

$$p_1 = \rho_{oo} g\eta e^{\int_y^o \frac{gdy}{c^2}} \tag{15}$$

We now consider the boundary conditions at the bottom (y = -h) and at the top (y = η) of the liquid. The bottom is supposed to be rigid, so that the vertical component of $\vec{U}$ is zero there:

$$v = 0 \quad \text{at} \quad y = -h. \tag{16}$$

The boundary condition at the free surface is that a particle of the fluid initially in the surface remains in the surface:

$$v = \frac{\partial\eta}{\partial t} + u\,\frac{\partial\eta}{\partial x} \quad \text{at} \quad y = \eta. \tag{17}$$

The term $u(\partial\eta/\partial x)$ on the right side of v is of second order, as is also the difference between v at $y = \eta$ and v at $y = 0$. Thus (17) can be written, to first order:

$$v = \frac{\partial\eta}{\partial t} \quad \text{at} \quad y = 0. \tag{18}$$

Eq. (9) can now be integrated with respect to y; with the help of the boundary condition (16), we obtain:

$$\rho_o v = -\int_{-h}^{y} \left(\frac{\partial\rho_1}{\partial t} + \rho_o \frac{\partial u}{\partial x}\right) dy.$$

Substitution from (4) and (15) gives:

$$\rho_o v = -\rho_{oo}\frac{\partial\eta}{\partial t}\left[e^{\int_{-h}^{o}\frac{gdy}{c^2}} - e^{\int_y^o \frac{gdy}{c^2}}\right] - \int_{-h}^{y} \rho_o \frac{\partial u}{\partial x}\, dy. \tag{19}$$

At y = 0, Eq. (19) becomes, with the help of (18):

$$\rho_{oo}\frac{\partial\eta}{\partial t} = -\rho_{oo}\frac{\partial\eta}{\partial t}\left[e^{\int_{-h}^{o}\frac{gdy}{c^2}} - 1\right] - \int_{-h}^{o} \rho_o \frac{\partial u}{\partial x}\, dy\,. \tag{20}$$

We now differentiate Eq. (20) with respect to t, and substitute from (8) and (15) to obtain:

$$0 = -\rho_{oo} \frac{\partial^2\eta}{\partial t^2} e^{\int_{-h}^{o} \frac{g dy}{c^2}} + \rho_{oo} g \frac{\partial^2\eta}{\partial x^2} \int_{-h}^{o} e^{\int_{y}^{o} \frac{g dy'}{c^2}} dy. \tag{21}$$

Eq. (21) has the usual wave equation form

$$\frac{\partial^2\eta}{\partial t^2} = c_w^{\,2} \frac{\partial^2\eta}{\partial x^2}, \tag{22}$$

where the wave speed c_w is given by:

$$c_w^{\,2} = g\, e^{-\int_{-h}^{o} \frac{g dy}{c^2}} \int_{-h}^{o} e^{\int_{y}^{o} \frac{g dy'}{c^2}} dy. \tag{23}$$

In the event that c (although not necessarily ρ_o) is independent of depth, Eq. (23) simplifies to:

$$c_w^{\,2} = c^2 \left(1 - e^{-gh/c^2}\right) \tag{24}$$

Now gh is the square of the speed c_g of gravitational waves in a shallow incompressible liquid. Thus c_w given by (24) is always less than both c_g and c. If $c \ll c_g$, the waves are mainly elastic and c_w is slightly less than c; if $c_g \ll c$, the waves are mainly gravitational and c_w is slightly less than c_g.

We must now return to the question of the neglect of the second integral on the right side of Eq. (13). In order to justify this neglect, we estimate the order of magnitude of the ratio of this integral to p_1 as given by (15). The integral is over-estimated by giving $\rho_o(\partial v/\partial t)$ its surface value $\rho_{oo}(\partial^2\eta/\partial t^2)$ and taking y equal to -h. Also, we can for this purpose assume that c is constant, and that

$$\frac{\partial^2\eta}{\partial t^2} = \omega^2\eta = \frac{4\pi^2 c_w^{\,2}\,\eta}{\lambda^2}$$

where ω and λ are the angular frequency and wave length of the (sinusoidal) wave motion of interest. With the help of (24), the ratio in question is thus estimated to be

$$\frac{4\pi^2 h^2}{\lambda^2} \; \frac{c^2}{gh} \; e^{-gh/c^2} \left(1 - e^{-gh/c^2}\right). \tag{25}$$

If the gravitational aspect of the wave motion dominates ($c_g \ll c$), the ratio (25) is small if $h^2 \ll (\lambda/2\pi)^2$. This is the condition that leads to the gravitational speed

$c_g = (gh)^{1/2}$ for a shallow incompressible liquid.[2] If, on the other hand, the elastic aspect dominates ($c \ll c_g$), the depth need not be small in comparison with the wave length in order for the above neglect to be justified. Thus Eqs. (22) and (23) are always valid for shallow liquids.

BOUNDARY CONDITIONS AT AN INTERFACE

An interface between two different liquids of the type considered above will be unstable unless the functions $\rho_0(y)$ are the same for the two liquids, so that the static pressures given by Eq. (12) are also the same at all depths. It may, however, be of some interest to consider in an approximate way the nature of the boundary conditions at an interface that is artificially maintained in some way. We can, for example, imagine that a vertical membrane separates the two liquids and supports the difference in static pressure, so that the two liquids need not even have the same depth. If at the same time the membrane is flexible enough to transmit pressure variations and horizontal velocity components (the restoring force and mass necessarily introduced by the membrane are ignored here), the interfacial boundary conditions are that p_1 and u (or $\partial u/\partial t$) be continuous. Then it is apparent from Eqs. (8) and (15) that waves of the type discussed above cannot be matched at all depths.

While a more exact (and far more complicated) treatment could be given if the detailed properties of the membrane were specified, we shall limit ourselves here to the simplest possible consideration. We assume that the gravitational aspect of the wave motion dominates, so that the exponential in (15) is close to unity for all depths. Then p_1 is continuous if $\rho_{00}\eta$ is continuous. We also assume that it is sufficient to make $\partial u/\partial t$ (and hence u) continuous at some average depth, at which $\rho_0(y)$ has the value ρ_a; then $(\rho_{00}/\rho_a)(\partial\eta/\partial x)$ must be continuous at the interface. It then follows from these assumptions that the quantity analogous to impedance, to be used in determining the effect of an interface, is $\rho_a c_w$.

[2] Coulson, C. A., "Waves", op. cit., pp. 62-66.